Using a desk calculator, though, to compute Eq. (42) one obtains

$$\alpha = 24.45°$$

which when used to compute S_0 and S_x [using Eqs. (43) and (44)] yields

$$S_0 = 59.95$$
$$S_x = 450.35$$

The results of Ref. 5 are

$$S_0 = 59.88$$
$$S_x = 450.45$$

It is concluded that the graphical method is accurate enough.

Conclusions

A new technique for designing a phase lead compensator on the S-plane has been presented in this work. This technique eliminates the need for trial and error cycles which are a feature of the present technique.[6-8] It can be carried out in part through a graphical construction or in an entirely analytical way. A digital computer program which implements the new technique can be written using the algorithm given by Eqs. (39–44).

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Vibration and Buckling of Shells under Initial Stresses

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The force-displacement matrix equations are formulated for a 48 degrees-of-freedom doubly curved shell finite element. The formulations include, in addition to stiffness, the effect of inertia force and initial stress. The formulation is achieved by a systematic integration method which utilizes the orderly patterns of the polynomial shape functions. Results include the study of free vibration of circular arches under hydrostatic pressure; cylindrical and paraboloidal shells subjected to uniform and linearly varying stresses acting in the middle surface. Particular attention is given to the influence of initial stress on the frequency of free vibration and the extrapolated stress corresponding to zero frequency which yields the static buckling stress. It is found in some examples that the relation between the square of the frequency and the initial stress is not linear for the case that the fundamental vibration mode shape is the same as the critical buckling mode.

Nomenclature

a, b, t	= length, width, and thickness of the element, respectively
D_m, D_f	= membrane and flexural rigidities, respectively
$f_i(\xi, \eta)$	= shape function associated with ith D.O.F.
$H(\xi), h(\alpha)$	= first-order Hermitian polynomial in ξ and α, respectively
m, n	= mode number of vibration
$[k], [m], [n]$	= element stiffness, mass, and incremental stiffness matrices, respectively
$\{p\}, \{q\}$	= element load and displacement vectors, respectively
R_1, R_2	= principal radii of curvature in ξ and η directions, respectively
r_1, r_2	= inverse of R_1 and R_2, respectively
S_1, S_2, S_{12}	= direct and shearing initial stresses in lb/in. ($S_{12} = S_3$)
u, v, w	= displacements in ξ, η, and z directions, respectively
α, β	= ξ/a and η/b, respectively
γ	= EI/EA for the arch element
$\varepsilon_1, \varepsilon_2, \varepsilon_{12}$	= direct and shearing strains in the middle surface of the shell
$\kappa_1, \kappa_2, \kappa_{12}$	= changes of direct and twisting curvatures
ξ, η, z	= curvilinear coordinate system
$\sigma_{\text{classical}}$	= $0.605\, Et/R$
ω	= frequency of vibration
$\lfloor\ \rfloor, \{\ \}, [\ ]$	= row, column, and rectangular matrices, respectively

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Introduction

THE widespread use of shells as structural components of flight vehicles has stimulated many investigators to study various aspects of the structural behavior of shells. The predictions of natural frequencies of vibration and the critical values of middle surface compressive stress of shell structures have attracted considerable attention. It is, however, quite common that the shell structures vibrate while they are subjected to certain initial stresses. Undoubtedly, a combined study of vibration of the shells in the presence of initial stresses will provide the structural engineers with more complete information than the analysis of vibration and stability separately.

The combined study of vibrations and initial stresses have been extensively treated for the cases of flat plates.[1-5] Because the equations of transverse motion for the case of flat plates are relatively simple, it was able to include quite complex middle-plane initial stresses such as direct stress,[1-5] shear stress,[4,5] and linearly varying stress.[5] Furthermore, complicated elastic boundary conditions were treated.[3]

In the case of shells, considerable complexity is involved in dealing with either the differential equations of motion or the energy expressions, the studies of the influence of initial stress on the vibration of shells have been confined to special cases of cylindrical shells subjected to uniform membrane stresses.[6-12]

In the early stage, various simplified versions of linear theory of motion were employed in studying the frequency of the radial mode of cylindrical shells due to the influence of initial, uniform, axial and circumferential stresses.[6-8] Later, Armenákas, and Herrmann[9] applied a linear bending theory to investigate the effect of initial uniform circumferential stress, uniform bending moment, and uniform radial shear on the vibration of an infinitely long cylindrical shell. By formulating the partial differential equations of motion with the inclusion of uniform initial stresses and then substituting the displacement functions into the equations of motion, they obtained the algebraic equations for the amplitude factors which yielded the nontrivial frequency equation. The bending theory was further extended by Armenákas[10] to treat the cylindrical shell with edges simply supported. By the use of Lagrangian equation, Koval[11] obtained the approximate frequency equations for the breathing mode vibrations of a clamped cylindrical shell subjected to uniform axial and circumferential stresses. Petyt[12] described four methods (extended Rayleigh-Ritz, triangular finite element, rectangular finite element, and Kantorovich methods) to treat the vibration of curved panels. A cylindrical panel subjected to uniform membrane stresses was treated.

It is seen in the previous treatments of the shells that only uniformly distributed membrane stresses were considered. The solutions to all of the examples seem to show that the relation between the membrane stress and the square of the frequency is linear in cases in which the vibration mode shape is the same as the buckling mode. This linear relationship is, however, not a general conclusion and the vibration must be analyzed at various stress levels. It will be interesting to investigate the effect of membrane stress which is not uniformly distributed in the shells so that the square of the frequency and the initial stress level exhibit a nonlinear relationship. It will also be useful to extend the treatment from cylindrical shells to more general cases which allow for double curvatures, irregular boundary shapes, complex supporting conditions, and even geometrical cut-outs, etc. To meet such objectives, a development in the finite element method seems quite appropriate.

In this paper, the linear stiffness and mass formulations developed previously[13-15] for a 48 degrees-of-freedom rectangular shell finite element are extended to include the nonlinear terms in the strain-displacement equations to formulate an incremental stiffness matrix. The complete element formulation is capable of analyzing the dynamic response of shells in the presence of distributed initial stresses. A systematic integration method is developed which enormously simplifies the element formulation. This method can account for nonuniform distribution of initial stress within an element.

The first example is a circular arch with both ends hinged or fixed. It is found that the square of the frequency varies linearly with the uniform radial pressure for the lowest antisymmetrical mode. The second example is a simply supported circular cylindrical shell. In the presence of uniform axial pressure, the square of the frequency is found, as previous workers did, linearly related to the pressure for the same fundamental mode of vibration and buckling. A pair of linearly varying stresses in the form of bending moments are then applied at both ends of the cylindrical shell. For the same fundamental mode of vibration and buckling, the interesting nonlinear relation between the square of the frequency and the end moment is found. A paraboloidal shell is finally treated with the inclusion of both uniform and linearly varying middle-surface stresses. In each example, the frequency of vibration with no initial stress and the buckling stress corresponding to zero frequency are evaluated through comparison with, whenever possible, alternative known solutions. All comparisons indicate reasonable agreement.

Description of a Shell Finite Element

A 48 degrees-of-freedom rectangular shell finite element is employed as an example to show the procedures for formulation and solution. The element is of constant thickness and has two constant radii of curvature (R_1 and R_2) corresponding to the curvilinear coordinates of the middle surface ξ and η, respectively. A detailed description of the element was given in Ref. 15.

It has been shown that not only the geometrical admissible conditions for the assembled set of elements were satisfied but also the six rigid-body modes were implicitly included.[13-15] The linear stiffness and mass matrices for this element have been formulated.[14,15] The formulation for the incremental stiffness matrix which requires the retention of second-order terms in the strain-displacement equations remains to be derived.

Because of the high number of degrees of freedom and the sophisticated displacement functions assumed for this element, the formulation of the incremental stiffness matrix is complex and lengthy. To overcome this difficulty, a systematic integration method is developed which makes the advantage of the patterns of the polynomial shape functions.

Based on this integration method, the assumed element degrees of freedom and displacement functions are systematically labelled. The nodal degrees of freedom are defined as

$$\underset{48\times 1}{\lfloor q \rfloor} = \lfloor u_1, u_{\xi_1}, u_{\eta_1}, u_{\xi\eta_1}, u_2, u_{\xi_2}, \ldots, u_{\xi\eta_4} \quad v_1, v_{\xi_1}, v_{\eta_1}, v_{\xi\eta_1}, v_2, v_{\xi_2}, \ldots, v_{\xi\eta_4} \quad w_1, w_{\xi_1}, \ldots, w_{\xi\eta_4} \rfloor \tag{1}$$

where ξ and η denote differentiations and the subscripts 1, 2, 3, and 4 denote nodal points at (0,0), (0,b), (a,0), and (a,b), respectively. The curvilinear coordinate axes ξ and η lie along edges 1–3 and 1–2, respectively.

The displacement functions are defined as

$$\begin{Bmatrix} u(\xi,\eta) \\ v(\xi,\eta) \\ w(\xi,\eta) \end{Bmatrix} = \begin{bmatrix} q_1, q_2, \ldots, q_{16} \\ q_{17}, q_{18}, \ldots, q_{32} \\ q_{33}, q_{34}, \ldots, q_{48} \end{bmatrix} \begin{Bmatrix} f_1(\xi,\eta) \\ f_2(\xi,\eta) \\ \vdots \\ f_{16}(\xi,\eta) \end{Bmatrix} \tag{2}$$

where $f_i(\xi,\eta)$ are the shape functions each of which is the cross product of two first-order Hermitian polynomials

$$f_i(\xi,\eta) = H_m(\xi)H_n(\eta) \tag{3}$$

in which m and n are integers ranging from 1 to 4. The relations among i, m, and n are given in Ref. 13.

Strain-Displacement Relations

Based on the Kirchhoff and Love assumptions for thin elastic shell, the strain components in a surface at a distance z from the middle-surface are defined as

$$\varepsilon_1{}^z = \varepsilon_1 - z\kappa_1, \quad \varepsilon_2{}^z = \varepsilon_2 - z\kappa_2, \quad \varepsilon_{12}{}^z = \varepsilon_{12} - z\kappa_{12} \tag{4}$$

The strains written in terms of displacements and their derivatives up to the complete quadratic terms were derived by Gallagher and Yang[16] for the formulation of incremental stiffness for a doubly curved shell finite element. Alternatively, a simpler set of nonlinear strain-displacement equations were obtained by Ogibalov[17] which differs from those given in Ref. 16 by 1) disregarding the terms containing the squares of curvatures

$1/R_1$ and $1/R_2$; 2) disregarding the second-order terms containing the derivatives of membrane displacements u and v; and 3) disregarding all the quadratic terms in the expressions for κ_1, κ_2, and κ_{12}. The equations suggested by Ogibalov are written as

$$
\begin{aligned}
\varepsilon_1 &= u_\xi + r_1 w + \tfrac{1}{2}(w_\xi)^2 - r_1(u)(w_\xi) \\
\varepsilon_2 &= v_\eta + r_2 w + \tfrac{1}{2}(w_\eta)^2 - r_2(v)(w_\eta) \\
\varepsilon_{12} &= v_\xi + u_\eta + (w_\xi)(w_\eta) - r_1(u)(w_\eta) - r_2(v)(w_\xi) \\
\kappa_1 &= r_1 u_\xi - w_{\xi\xi} \\
\kappa_2 &= r_2 v_\eta - w_{\eta\eta} \\
\kappa_{12} &= r_2 v_\xi + r_1 u_\eta - 2w_{\xi\eta}
\end{aligned}
\tag{5}
$$

where $r_1 = 1/R_1$ and $r_2 = 1/R_2$. This set of equations was shown by Ogivalov to give sufficiently accurate solutions to the stability and large deflection problems of thin elastic shell in the absence of shearing strains. It is thus adopted for this study.

Element Formulation

The stiffness formulation for the vibration problem for a shell element under initial stress can be obtained from the Lagrangian equation. By a procedure similar to that for the case of plate,[5] the Lagrangian function for a shell finite element can be obtained as

$$
L = \tfrac{1}{2}\lfloor \dot{q} \rfloor [m]\{\dot{q}\} + \tfrac{1}{2}\lfloor q \rfloor [k]\{q\} + \sum_{i=1}^{3} (D_m/2) \int_A \lfloor q \rfloor \{g_i\} \times \lfloor q \rfloor [B_i]\{q\}\, dA - \lfloor p \rfloor \{q\} \tag{6}
$$

The mass matrix $[m]$ and stiffness matrix $[k]$ are well-known.[14,15] In the third expression of Eq. (6)

$$
\underset{48\times 1}{\{g_1\}} = \begin{Bmatrix} f_\xi \\ \nu f_\eta \\ (r_1 + \nu r_2) f \end{Bmatrix}; \quad
\underset{48+1}{\{g_2\}} = \begin{Bmatrix} \nu f_\xi \\ f_\eta \\ (\nu r_1 + r_2) f \end{Bmatrix};
$$

$$
\underset{48\times 1}{\{g_3\}} = \frac{1-\nu}{2} \begin{Bmatrix} f_\eta \\ f_\xi \\ 0 \end{Bmatrix} \tag{7}
$$

and

$$
\underset{48\times 48}{[B_1]} = \begin{bmatrix} B_{uu} & B_{uv} & B_{uw} \\ B_{vu} & B_{vv} & B_{vw} \\ B_{wu} & B_{wv} & B_{ww} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -r_1 f f_\xi \\ 0 & 0 & 0 \\ -r_1 f_\xi f & 0 & f_\xi f_\xi \end{bmatrix}
$$

$$
\underset{48\times 48}{[B_2]} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -r_2 f f_\eta \\ 0 & -r_2 f_\eta f & f_\eta f_\eta \end{bmatrix};
$$

$$
\underset{48\times 48}{[B_3]} = \begin{bmatrix} 0 & 0 & -r_1 f f_\eta \\ 0 & 0 & -r_2 f f_\xi \\ -r_1 f_\eta f & -r_2 f_\xi f & f_\xi f_\eta + f_\eta f_\xi \end{bmatrix} \tag{8}
$$

For the case of membrane theory, the shells respond to the membrane load in a linear membrane behavior. Substituting the constant stress resultants for the displacements into the third expression in Eq. (6), the expression becomes

$$
\sum_{i=1}^{3} (S_i/2) \int_A \lfloor q \rfloor [B]\{q\}\, dA \tag{9}
$$

The equations of homonic motion can thus be obtained by Lagrangian equation

$$
\{p\} = [-\omega^2[m] + [k] + [n]]\{q\} \tag{10}
$$

with

$$
[n] = \sum_{i=1}^{3} S_i \int_A [B_i]\, dA \tag{11}
$$

Since the membrane and flexural behaviors are not coupled, Eq. (10) can be separated into two parts. The flexural part constitutes a set of eigenvalue equations because the flexural loads are zeros in the membrane theory.

Table 1 Relations between i and (m, n, q, r) defined in Eq. (15)

	$f_i(\xi,\eta)$				$f_{\xi_i}(\xi,\eta)$				$f_{\eta_i}(\xi,\eta)$				$f_{\xi\xi_i}(\xi,\eta)$				$f_{\eta\eta_i}(\xi,\eta)$				$f_{\xi\eta_i}(\xi,\eta)$			
i	m	n	q	r	m	n	q	r	m	n	q	r	m	n	q	r	m	n	q	r	m	n	q	r
1	1	1	0	0	5	1	−1	0	1	5	0	−1	9	1	−2	0	1	9	0	−2	5	5	−1	−1
2	3	1	1	0	7	1	0	0	3	5	1	−1	11	1	−1	0	3	9	1	−2	7	5	0	−1
3	1	3	0	1	5	3	−1	1	1	7	0	0	9	3	−2	1	1	11	0	−1	5	7	−1	0
4	3	3	1	1	7	3	0	1	3	7	1	0	11	3	−1	1	3	11	1	−1	7	7	0	0
5	1	2	0	0	5	2	−1	0	1	6	0	−1	9	2	−2	0	1	10	0	−2	5	6	−1	−1
6	3	2	1	0	7	2	0	0	3	6	1	−1	11	2	−1	0	3	10	1	−2	7	6	0	−1
7	1	4	0	1	5	4	−1	1	1	8	0	0	9	4	−2	1	1	12	0	−1	5	8	−1	0
8	3	4	1	1	7	4	0	1	3	8	1	0	11	4	−1	1	3	12	1	−1	7	8	0	0
9	2	1	0	0	6	1	−1	0	2	5	0	−1	10	1	−2	0	2	9	0	−2	6	5	−1	−1
10	4	1	1	0	8	1	0	0	4	5	1	−1	12	1	−1	0	4	9	1	−2	8	5	0	−1
11	2	3	0	1	6	3	−1	1	2	7	0	0	10	3	−2	1	2	11	0	−1	6	7	−1	0
12	4	3	1	1	8	3	0	1	4	7	1	0	12	3	−1	1	4	11	1	−1	8	7	0	0
13	2	2	0	0	6	2	−1	0	2	6	0	−1	10	2	−2	0	2	10	0	−2	6	6	−1	−1
14	4	2	1	0	8	2	0	0	4	6	1	−1	12	2	−1	0	4	10	1	−2	8	6	0	−1
15	2	4	0	1	6	4	−1	1	2	8	0	0	10	4	−2	1	2	12	0	−1	6	8	−1	0
16	4	4	1	1	8	4	0	1	4	8	1	0	12	4	−1	1	4	12	1	−1	8	8	0	0

In general, however, the shells do not always respond to the external load in a linear membrane behavior. The membrane and flexural behaviors are coupled. Thus the energy expression (9) must be written in terms of displacements rather than constant membrane stresses. The application of Lagrangian equation then yields

$$
\{p\} = \left[-\omega^2[m] + [k] + \frac{3D_m}{2} \sum_{i=1}^{3} \int_A [B_i] \lfloor q \rfloor \{g_i\}\, dA \right]\{q\} \tag{12}
$$

Equation (12) is not an eigenvalue formulation because the loads $\{p\}$ are not all zeros. Equation (12) must then be subjected to an incremental operation

$$
\{\delta p\} = [-\omega^2[m] + [k] + [n']]\{\delta q\} \tag{13}
$$

where

$$
[n'] = 3D_m \sum_{i=1}^{3} \int_A [B_i] \lfloor q \rfloor \{g_i\}\, dA \tag{14}
$$

For any level of loadings, the increments $\{\delta p\}$ vanish. Equation (13) can then be solved for eigenvalues.

Integration Method

The shape functions $f_i(\xi, \eta)$ given in Eq. (3) and their derivatives may be written in a factorized form

$$
f_i(\xi, \eta) = H_m(\xi) H_n(\eta) = a^q b^r h_m(\alpha) h_n(\beta) \tag{15}
$$

where α and β are the nondimensional variables with $\alpha = \xi/a$ and $\beta = \eta/b$. Thus $h(\alpha)$ and $h(\beta)$ are the first-order Hermitian polynomials of dimensionless parameters α and β. Based on Eq. (15), the first and second derivatives of functions $f(\xi, \eta)$ with respect to ξ and η can be obtained. The relations between i and (m, n, q, r) are given in Table 1. By such nondimensionalization, the integration of an element with area of a by b can be changed to that of 1 by 1.

The integration of products of functions h in Eq. (15) over the element area can be written in the form that

$$
C(i, j) = \int_0^1 h_i(\alpha) h_j(\alpha)\, d\alpha = \int_0^1 h_i(\beta) h_j(\beta)\, d\beta \tag{16}
$$

There are a total of 12 h-functions. Thus the matrix $C(i, j)$ contains 12×12 constants. Due to symmetry, only 78 constants need be found which are given in Table 2.

The element incremental stiffness matrix are defined in Eq. (11) or (14). The explicit form of the coefficients in the matrix can be obtained by using Tables 1 and 2. To give an example, the coefficient $n(37, 30)$ is presented. Since the matrix is divided into nine submatrices, the coefficient at 37th row and the 30th column of the matrix is the coefficient at 5th row and 14th column in the submatrix related to the w and v displacements.

Table 2 Values for $C(i,j)$ defined in Eq. (16)

i	j	$C(i,j)$	i	j	$C(i,j)$	i	j	$C(i,j)$
1	1	13/ 35	7	6	−1/10	10	8	−1/ 1
2	1	9/ 70	7	7	2/15	10	9	−12/ 1
2	2	13/ 35	8	1	−1/10	10	10	12/ 1
3	1	11/210	8	2	1/10	11	1	−11/10
3	2	13/420	8	3	−1/60	11	2	1/10
3	3	1/105	8	4	0/ 1	11	3	−2/15
4	1	−13/420	8	5	1/10	11	4	1/30
4	2	−11/210	8	6	−1/10	11	5	1/ 1
4	3	−1/140	8	7	−1/30	11	6	−1/ 1
4	4	1/105	8	8	2/15	11	7	−1/ 2
5	1	−1/ 2	9	1	−6/ 5	11	8	1/ 2
5	2	−1/ 2	9	2	6/ 5	11	9	6/ 1
5	3	−1/ 10	9	3	−1/10	11	10	−6/ 1
5	4	1/ 10	9	4	−1/10	11	11	4/ 1
5	5	6/ 5	9	5	0/ 1	12	1	−1/10
6	1	1/ 2	9	6	0/ 1	12	2	11/10
6	2	1/ 2	9	7	−1/ 1	12	3	1/30
6	3	1/ 10	9	8	1/ 1	12	4	−2/15
6	4	−1/ 10	9	9	12/ 1	12	5	−1/ 1
6	5	−6/ 5	10	1	6/ 5	12	6	1/ 1
6	6	6/ 5	10	2	−6/ 5	12	7	−1/ 2
7	1	1/ 10	10	3	1/10	12	8	1/ 2
7	2	−1/ 10	10	4	1/10	12	9	6/ 1
7	3	0/ 1	10	5	0/ 1	12	10	−6/ 1
7	4	1/ 60	10	6	0/ 1	12	11	2/ 1
7	5	1/ 10	10	7	1/ 1	12	12	4/ 1

From Eqs. (11) and (8), it is obtained that

$$n_{wv}(5,14) = S_1(0) - S_2\int_A r_2 f_{\eta_5} f_{14}\, dA - S_{12}\int_A r_2 f_{\xi_5} f_{14}\, dA \qquad (17)$$

where $dA = ab\,d\alpha\, d\beta$; the shape functions f and their derivatives are defined in Eq. (15). From Table 1 in the columns under $f_{\xi_i}(\xi,\eta)$ with $i = 5$, the indices m, n, q, and r are found. Thus from Eq. (15) and Table 1

$$f_{\xi_5} = a^{-1} b^0 h_5(\alpha) h_2(\beta) \qquad (18)$$

By the same way

$$f_{\eta_5} = b^{-1} h_1(\alpha) h_6(\beta); \qquad f_{14} = a h_4(\alpha) h_2(\beta) \qquad (19)$$

Substituting the f-functions into Eq. (17) and defining $C(i,j)$ by Eq. (16) yield

$$n_{wv}(5,14) = -S_2 a^2 r_2 C(1,4) C(2,6) - S_{12} ab r_2 C(4,5) C(2,2) \qquad (20)$$

The values for $C(i,j)$ are given by Table 2. Finally

$$n_{wv}(5,14) = \tfrac{13}{840} a^2 r_2 S_2 - \tfrac{13}{350} ab r_2 S_{12} \qquad (21)$$

This procedure for obtaining the explicit coefficients in the incremental stiffness matrix can be written in a Fortran subroutine with Table 1 as input while Table 2 is generated by the computer. This method is particularly suitable for treating nonuniformly distributed initial stress. An example of incorporating linearly varying initial stress within an element will be given in the next section.

In this study, the element stiffness and mass matrices are also formulated by this integration method. Details are given in Ref. 18.

Results

a) Circular Arch Subjected to Uniform Radial Pressure

When the radius of curvature R_2 becomes infinity and the dimension in the η direction becomes unity, the present shell finite element becomes a circular arch element. In this particular case, only two nodal points and eight degrees of freedom remain

$$\lfloor q \rfloor = \lfloor u_1, u_2, u_{\xi_1}, u_{\xi_2}, w_1, w_2, w_{\xi_1}, w_{\xi_2} \rfloor \qquad (22)$$

The displacement functions given in Eq. (2) are simplified

$$u(\xi) = f_1 u_1 + f_2 u_2 + f_3 u_{\xi_1} + f_4 u_{\xi_2}$$
$$w(\xi) = f_1 w_1 + f_2 w_2 + f_3 w_{\xi_1} + f_4 w_{\xi_2} \qquad (23)$$

The middle-surface strain-displacement relations of Eq. (5) become

$$\varepsilon_1 = u_\xi + r_1 w + \tfrac{1}{2}(w_\xi)^2 - r_1(u)(w_\xi)$$
$$\kappa_1 = r_1 u_\xi - w_{\xi\xi} \qquad (24)$$

Thus the element formulations become simplified

$$[m] = \rho A \int_l \begin{bmatrix} ff & 0 \\ 0 & ff \end{bmatrix} d\xi$$

$$[k] = EA \int_l \begin{bmatrix} (1+\gamma r_1{}^2) f_\xi f_\xi & r_1 f_\xi f - r_1 \gamma f_\xi f_{\xi\xi} \\ r_1 f f_\xi - r_1 \gamma f_{\xi\xi} f_\xi & r_1{}^2 ff + \gamma f_{\xi\xi} f_{\xi\xi} \end{bmatrix} d\xi \qquad (25)$$

$$[n] = S_1 \int_l \begin{bmatrix} r_1{}^2 ff & -r_1 f f_\xi \\ -r_1 f_\xi f & f_\xi f_\xi \end{bmatrix} d\xi$$

A 120° circular arch with ends either both fixed or both hinged was chosen. The arch was subjected to uniform inward radial pressure. The results for the lowest frequency factor were obtained for increasing pressure by using 12 elements. They were plotted in Fig. 1. For both end conditions, the square of the frequency for the lowest antisymmetrical mode is seen to relate linearly to the pressure level.

For the case of no pressure, the frequency was obtained previously by Wolf[19] by the use of straight elements. The dimensionless fundamental frequency $(\omega^2 L^4 \rho A/EI)^{1/2}$ obtained by Wolf was 30.36 for the hinged arch and 51.92 for the fixed arch. The present solution yielded 30.36 for the hinged arch and 51.93 for the fixed arch. The two solutions are virtually identical.

For the extrapolated buckling value of pressure, a comparative basis is available in Ref. 20. According to Ref. 20, the dimensionless buckling pressure (pR^3/EI) was 18.2 for the fixed arch and 8 for the hinged arch. The present analysis gave values of 19.3 for the fixed arch and 8.5 for the hinged arch. It is noted that inextensibility was assumed in Ref. 20 while in the present study, a finite value of $EA = 1200EI$ was assumed.

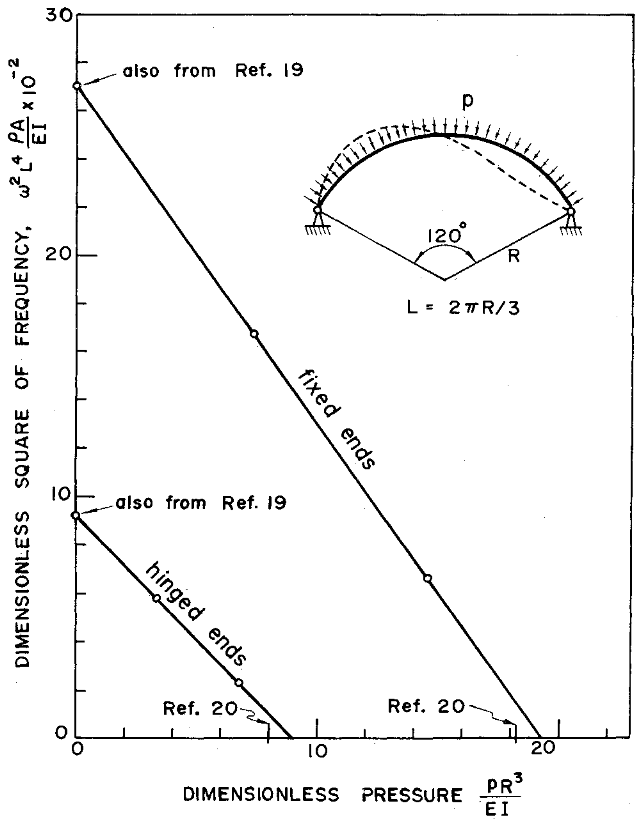

Fig. 1 Square of frequency vs pressure for the lowest antisymmetrical mode of a circular arch.

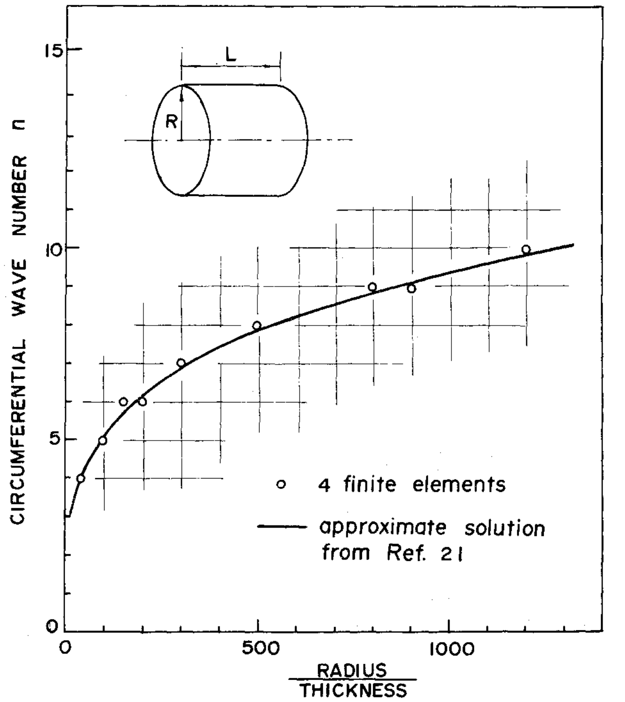

Fig. 2 Circumferential wave numbers associated with the fundamental modes for cylindrical shell ($m = 1$; $\nu = 0.3$).

b) Simply-Supported Circular Cylindrical Shell Subjected to Uniform Axial Pressure

For the vibrational analysis of simply supported circular cylindrical shell, displacement functions satisfying the boundary conditions may be assumed as

$$\begin{aligned} u &= \bar{u}_{mn} \cos(m\pi\xi/L) \sin n\theta \, e^{i\omega t} \\ v &= \bar{v}_{mn} \sin(m\pi\xi/L) \cos n\theta \, e^{i\omega t} \\ w &= \bar{w}_{mn} \sin(m\pi\xi/L) \sin n\theta \, e^{i\omega t} \end{aligned} \tag{26}$$

where $\bar{u}$, $\bar{v}$, and $\bar{w}$ are amplitudes; m and n are the half longitudinal and full circumferential wave numbers, respectively; and θ is the central angle. For the fundamental mode, m is assumed to be one and n remains to be found.

In this study, a typical portion bounded by $\xi = 0$, $\xi = L/2m$, $\eta = 0$, and $\eta = \pi R/2n$ was considered and a 2 by 2 mesh was used. Based on the displacement assumption of Eq. (26) and the assumption that $m = 1$, the boundary conditions are specified as:

$$\begin{aligned} u_\xi = u_{\xi\eta} = v = v_\eta = w = w_\eta &= 0 \quad \text{at} \quad \xi = 0 \\ u = u_\eta = v_\xi = v_{\xi\eta} = w_\xi = w_{\xi\eta} &= 0 \quad \text{at} \quad \xi = L/2 \\ u = u_\xi = v_\eta = v_{\xi\eta} = w = w_\xi &= 0 \quad \text{at} \quad \eta = 0 \\ u_\eta = u_{\xi\eta} = v = v_\xi = w_\eta = w_{\xi\eta} &= 0 \quad \text{at} \quad \eta = \pi R/2n \end{aligned} \tag{27}$$

The first analysis was made on the study of fundamental circumferential mode shapes for a cylinder with length-radius ratio (L/R) of 2 and various radius-thickness ratios (R/t). For each value of R/t, various integer values of n were assumed and the corresponding frequencies were found. The n-number that causes the minimum value of frequency is the circumferential wave number for that R/t value. The results were presented in Fig. 2. The frequency equations for finding the values of n can also be approximated by substituting the w-displacement function given by Eq. (25) into Donnell's eighth-order differential equation.[21] Such results were also shown in Fig. 2. Good agreement is indicated. It is noted that the present boundary conditions assumed by Eq. (27) were conforming to the w-displacement function assumed in Ref. 21 so that the comparison is possible.

A cylinder with $L/R = 2$ and $R/t = 500$ was then considered. According to Fig. 2, the value of n is 8 for the case of free vibration. It was expected that when subjected to uniform axial compression, the shell should buckle in the same mode shape as the fundamental mode of vibration. With $m = 1$, it was found that for $n = 6, 7, 8, 9$ and 10, the critical buckling stresses were 1.194, 1.145, 1.035, 1.162, and 1.53 times of the classical buckling stress, respectively. The number of n that causes the smallest buckling stress is 8 which is the same as that found in the fundamental mode of vibration.

For the modes of $m = 1$ and $n = 8$, the fundamental frequencies were found for various pressures and the results were given in Fig. 3. The relation between the square of frequency and the pressure is seen to be linear. The frequency was found to be 1.028 times of that obtained by Voss.[22] The extrapolated buckling stress was found to be 1.035 times of that from classical solution.

c) Simply-Supported Cylindrical Shell Subjected to Bending Moments at the Ends

The cylinder considered in this case is the same as that of the previous case. The bending moments were applied in the form of linearly varying membrane stresses as shown in Fig. 3. Because of the antisymmetrical distribution of the bending stresses, a quarter of the shell was considered and a mesh of 1 by 16 was used. The initial stresses were approximated by a step representation so that the initial stress was constant within each element.

The results were obtained for the fundamental frequencies ($m = 1$ and $n = 8$) for various levels of bending moment and shown in Fig. 3. It is interesting to find that the relationship between the square of the frequency and the level of the bending load is not linear. The maximum value of extrapolated buckling bending stress was 0.829 Et/R which is 1.37 times of the classical buckling value for uniformly compressed cylinder.

The buckling problem of the subject cylinder was studied previously by Seide and Weingarten[23] by using the Galerkin's method. The Galerkin's method was applied on the basis of Batdorf's modified version of Donnell's equation and the series assumption of the radial displacement that

$$w = \sin \frac{m\pi\xi}{L} \sum_{n=0}^{\infty} a_n \cos n\theta \tag{28}$$

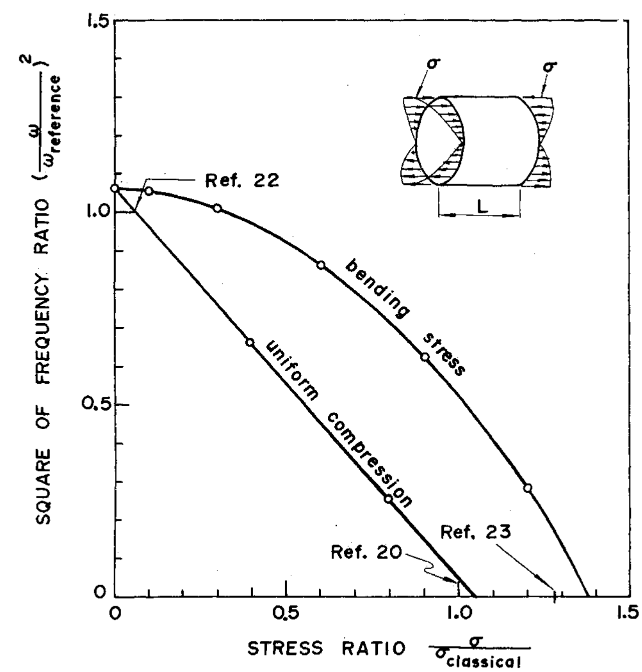

Fig. 3 Square of frequency vs initial stress for cylindrical shell ($m = 1$; $n = 8$; $R/L = 0.5$; $R/t = 500$; $\nu = 0.3$).

A table was provided in Ref. 23 which gave the maximum value of linearly varying buckling stress for cylinders with various geometries. Although the solution for the present cylinder is not available in that table, it can easily be obtained by following the procedure outlined in Ref. 23.

The maximum value of buckling bending stress was found to be $0.774\,Et/R$ which is 1.27 times of the buckling value for uniformly compressed cylinder. When Eq. (28) was used in conjunction with Galerkin's method, the longitudinal half wave number was set to be 1 and the circumferential mode shape was set to be the summation of 30 modes with $n = 1$ to 30. The normalized amplitudes a_n for the circumferential mode were found as

$$\lfloor a_1, a_2, \ldots, a_{30} \rfloor \approx \lfloor 10^{-8}, 10^{-6}, 10^{-4}, 10^{-3}, 0.084, 0.36, 0.793\ 1, 0.775, 0.392, 0.136, 0.033, 10^{-3}, 10^{-4}\ 10^{-5}, 10^{-6}, 10^{-7}, 10^{-8} \ldots \rfloor \qquad (29)$$

It is seen in Eq. (29) that the superposition of 30 different circumferential modes was dominated by the 8th mode and its adjacent modes. Therefore, it seems not unfair to compare the present extrapolated maximum buckling stress to that obtained by the Galerkin's method. The former is higher than the latter by 7.9%.

d) Simply-Supported Paraboloidal Shell with Rectangular Base

The equation of middle-surface of the shell is described by

$$z = H - H/2(x^2/l_1{}^2 + y^2/l_2{}^2) \qquad (30)$$

where the rise $H = 1$ in.; the lengths $l_1 = l_2 = l = 10$ in.; $|x| \leqq l_1$; and $|y| \leqq l_2$; $R_1 = R_2 = 100$ in.; and $t = 0.5$ in. This shell was considered by Reissner[24] for the simply-supported case and the frequency equation was obtained.

In this study, a quadrant of the shell was considered. The boundary conditions were determined on the basis of Reissner's displacement function and Airy's stress function whose derivatives were expressed in terms of displacements. Thus

$$\begin{aligned} u = u_y = v_x = v_{xy} = w_x = w_{xy} = 0 \quad &\text{at} \quad x = 0 \\ u_x = u_{xy} = v = v_y = w = w_y = 0 \quad &\text{at} \quad x = l_1 \end{aligned} \qquad (31)$$

The boundaries at $y = 0$ and $y = l_2$ had similar conditions.

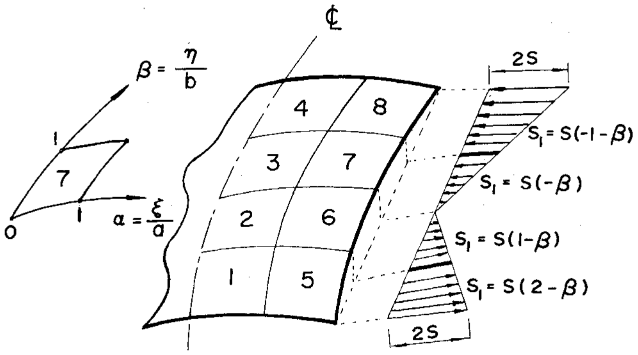

Fig. 5 Linearly varying initial stress on the edge of paraboloidal shell (β is the local coordinate for each element).

The fundamental frequencies were then obtained for different levels of uniform edge compression both uniaxially and biaxially. The results are shown in Fig. 4. It is seen that the relation between the square of the frequency and the level of initial stress is linear for both cases. For the case of biaxial compression, the extrapolated buckling stress was found to be 1.04 times of that ($0.605\,Et/R$) obtained by Haydl.[25] The buckling stress for the case of biaxial compression is not directly available from Ref. 25. It was found here to be twice of that in the uniaxial case.

The next example was to investigate the case that the uniaxial stresses were distributed linearly in the form of a pair of bending moment as shown in Fig. 4. Because of antisymmetry, only half of the shell was considered. The element idealization and the initial stress distribution are shown in Fig. 5. In deriving the incremental stiffness matrix, an additional pattern of integration was involved. That is

$$C'(i,j) = \int_0^1 \alpha h_i(\alpha) h_j(\alpha)\, d\alpha = \int_0^1 \beta h_i(\beta) h_j(\beta)\, d\beta \qquad (32)$$

The explicit presentation of $C'(i,j)$ is given in Table 3. Thus, if

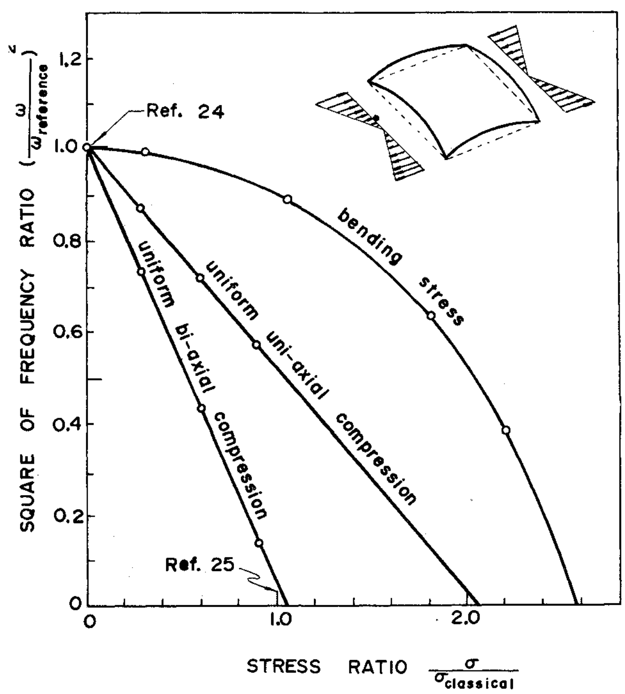

Fig. 4 Square of frequency vs initial stress for paraboloidal shell for the fundamental mode ($\nu = 0.3$).

Table 3 Values for $C'(i,j)$ defined in Eq. (32)

i	j	$C'(i,j)$	i	j	$C'(i,j)$	i	j	$C'(i,j)$
1	1	3/35	7	6	−1/10	10	8	−7/10
2	1	9/140	7	7	1/30	10	9	−6/1
2	2	2/7	8	1	−11/420	10	10	6/1
3	1	1/60	8	2	23/210	11	1	−1/5
3	2	1/60	8	3	−1/210	11	2	1/5
3	3	1/280	8	4	−1/210	11	3	−1/30
4	1	−1/70	8	5	0/1	11	4	0/1
4	2	−1/28	8	6	0/1	11	5	1/5
4	3	−1/280	8	7	−1/60	11	6	−1/5
4	4	1/168	8	8	1/10	11	7	−1/15
5	1	−13/70	9	1	−1/10	11	8	4/15
5	2	−11/35	9	2	11/10	11	9	2/1
5	3	−3/70	9	3	0/1	11	10	−2/1
5	4	2/35	9	4	−1/10	11	11	1/1
5	5	3/5	9	5	−3/5	12	1	1/10
6	1	13/70	9	6	3/5	12	2	9/10
6	2	11/35	9	7	−3/10	12	3	1/30
6	3	3/70	9	8	7/10	12	4	−1/10
6	4	−2/35	9	9	6/1	12	5	−4/5
6	5	−3/5	10	1	1/10	12	6	4/5
6	6	3/5	10	2	−11/10	12	7	−7/30
7	1	−1/105	10	3	0/1	12	8	13/30
7	2	−31/420	10	4	1/10	12	9	4/1
7	3	−1/210	10	5	3/5	12	10	−4/1
7	4	1/84	10	6	−3/5	12	11	1/1
7	5	1/10	10	7	3/10	12	12	3/1

the distribution of initial stress is of square or cubic variation, it is merely necessary to obtain one or two more tables related to the integration of $\alpha^2 hh$ or $\alpha^3 hh$, respectively.

The relation between the square of the frequency and the level of bending stress for the fundamental mode was found and shown in Fig. 4. The interesting nonlinear relation is seen. An alternative solution for buckling stress seems not, however, available in existing literature for comparison.

Conclusions

The stiffness and mass formulations for a 48 D.O.F. shell element has been extended to include the effect of uniform and linearly varying initial stresses. The formulations were achieved by a systematic integration method which utilizes the orderly patterns of polynomial shape functions and nondimensionalizes the integration limit.

Combined investigations of vibration and stability have been performed for the examples of arches, cylindrical shells, and paraboloidal shells. In each example, results have been presented for the influence of initial stress on the fundamental frequency. In some cases, the influence presents a nonlinear phenomenon.

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